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STATISTICS/ECONOMETRICS PREP COURSE – PROF. MASSIMO GUIDOLIN

SECOND PART, LECTURE 4: CONFIDENCE INTERVALS

OVERVIEW

1) General notion – sample (random) intervals
2) Coverage and confidence of intervals
3) Finding confidence intervals by inversion of (non-) rejection regions of tests
4) A few examples

From Point to Set (Interval) Inferences

- Point estimation of a parameter θ boils down inference to a guess of a single value as the value of θ
- In most situations, it seems more plausible to provide a guess in the form of an interval
- Inference in a set problem is the statement that "$\theta \in \mathbb{C}$" where $\mathbb{C} \subset \Theta$ and $\mathbb{C}$ = C(**x**) is determined by the sample of the data
  - If θ is real-valued, then we usually prefer the set estimate $\mathbb{C}$ to be an interval
- Definition: An interval estimate of a real-valued parameter θ is any pair of functions, $L(X_1, ..., X_n)$ and $U(X_1, ..., X_n)$ of a sample that satisfy $L(\mathbf{X}) \leq U(\mathbf{X})$; if **x** is observed, the inference $L(\mathbf{x}) \leq \theta \leq U(\mathbf{x})$ is made and the random interval $[L(\mathbf{x}), U(\mathbf{x})]$ is called an interval estimator
  - Although in the majority of cases we will work with finite values for

FROM POINT TO SET (INTERVAL) INFERENCES

L and U, there is sometimes interest in one-sided interval estimates

- In the definition, using an open or closed interval, [L(**x**), U(**x**)], is immaterial

- What is the gain from using interval estimators? On the one hand, we have given up precision, so there must be some gain
- The answer is that we benefit from some confidence or assurance as to our assertions concerning an estimate
  - Let's stop and think: we know that the sample mean is MLE and as such unbiased, consistent, asymptotically efficient, etc. Great, but what is Pr(sample mean = μ)? The answer is Pr(sample mean = μ) = 0!
    - Therefore one needs to broaden the interval in order to obtain a stronger endorsement, e.g., in the case of an IID N(μ,σ^2) population

$$\Pr\left(\mu \in [\bar{X}-a, \bar{X}+a]\right) = \Pr\left(\bar{X}-a \le \mu \le \bar{X}+a\right) = \Pr\left(-a \le \bar{X}-\mu \le a\right)$$

$$= \Pr\left(-\frac{a}{S/\sqrt{n}} \le \frac{\bar{X}-\mu}{S/\sqrt{n}} \le \frac{a}{S/\sqrt{n}}\right) = \Pr\left(-\frac{a}{S/\sqrt{n}} \le T_{n-1} \le \frac{a}{S/\sqrt{n}}\right)$$

Coverage and Confidence of Intervals

  - Sets or intervals? In general one estimates confidence sets, but in finance most of the time one looks for intervals, CIs
- What confidence or assurance?
- Definition [COVERAGE]: For an interval estimator $[L(\mathbf{X}), U(\mathbf{X})]$ of a parameter θ, the coverage probability is the probability that the random interval $[L(\mathbf{X}), U(\mathbf{X})]$ covers the true parameter, θ, or $\Pr(\theta \in [L(\mathbf{X}), U(\mathbf{X})])$
- Definition [CONFIDENCE]: For an interval estimator $[L(\mathbf{X}), U(\mathbf{X})]$ of a parameter θ, the confidence coefficient is the infimum of the coverage probabilities, $\inf_\theta \Pr(\theta \in [L(\mathbf{X}), U(\mathbf{X})])$
- CIs are for the random sample, not for the parameter θ: when we write statements such as $\Pr(\theta \in [L(\mathbf{X}), U(\mathbf{X})])$, these probability statements refer to $\mathbf{X}$, not θ
  - Re-write $\Pr(\theta \in [L(\mathbf{X}), U(\mathbf{X})])$ as $\Pr(L(\mathbf{X}) \leq \theta)$ together with $\Pr(U(\mathbf{X}) \geq \theta)$

Methods to Find Confidence Intervals

  - Since we do not know the true value of θ, we can only guarantee a coverage probability equal to the infimum, the confidence coefficient
- Now the good news: There is a strong correspondence between hypothesis testing and interval estimation: in general, every confidence set corresponds to a test and vice versa
- So you almost, almost, do not need to separately study this...
  - Example: Let $X_1, \ldots, X_n$ be IID $N(\mu,\sigma^2)$ and consider testing $H_0: \mu = \mu_o$ versus $H_1: \mu \neq \mu_o$ with σ known. For a fixed α level, a reasonable test (in fact, the MPU) has rejection region $\{\mathbf{x}: |\bar{x} - \mu_0| > z_{\alpha/2}\sigma/\sqrt{n}\}$ which implies that the null is not rejected for sample points with

$$\{\mathbf{x}: z_{\alpha/2}\sigma/\sqrt{n} \leq \bar{x} - \mu_0 \leq z_{\alpha/2}\sigma/\sqrt{n}\} \Longleftrightarrow \bar{x} - z_{\alpha/2}\sigma/\sqrt{n} \leq \mu_0 \leq \bar{x} + z_{\alpha/2}\sigma/\sqrt{n}$$

Since the test has size α, this means that $\Pr(H_o \text{ is not rejected} | \mu = \mu_0) = 1 - \alpha$. Combining this with the characterization, we can write

$$\Pr\left(\bar{x} - z_{\alpha/2}\sigma/\sqrt{n} \leq \mu_0 \leq \bar{x} + z_{\alpha/2}\sigma/\sqrt{n}\right) = 1 - \alpha$$

METHODS TO FIND CONFIDENCE INTERVALS

– This interval is obtained by inverting the (complement of the) rejection region of the level α test
– There is no guarantee that the confidence set obtained by test inversion will be an interval
  - In most cases, however, one-sided tests give one-sided intervals, two-sided tests give two-sided intervals, strange-shaped rejection regions give strange-shaped confidence sets
– <u>Example</u>: As another example, we show how do you go about inverting an LRT. Suppose we want a CI for the mean, λ, of an exponential(λ) population, with PDF $f(x; \lambda)=(1/\lambda)\exp(-x/\lambda)$. We can obtain such an interval by inverting a size α test of $H_o: \lambda = \lambda_0$ versus $H_1: \lambda \neq \lambda_0$.
– If we take a random sample $X_1, \ldots, X_n$, the LRT statistic is given by

$$LRT(\mathbf{x}) = \frac{L(\lambda_0; \mathbf{x})}{\sup_\lambda L(\theta; \mathbf{x})} = \frac{\frac{1}{\lambda_0^n}\exp\left(-\frac{1}{\lambda_0}\sum_{i=1}^n x_i\right)}{\sup_\lambda \frac{1}{\bar{x}^n}\exp(-n)} = \left(\frac{\bar{x}}{\lambda_0}\right)^n e^n \exp\left(1 - \frac{\bar{x}}{\lambda_0}\right)$$

– For fixed λ_0, the non-rejection region is given by

Methods to Find Confidence Intervals

$$\left\{\mathbf{x} : \left(\frac{\bar{x}}{\lambda_0}\right)^n \exp\left(1 - \frac{\bar{x}}{\lambda_0}\right) \geq k^*\right\} \quad (e^n \text{ absorbed into } k^*)$$

where k* is a constant chosen to satisfy Pr{x ∈ not reject} = 1-α

– Inverting this region gives the 1-α set

$$C(\mathbf{x}) = \left\{\lambda : \left(\frac{\bar{x}}{\lambda}\right)^n \exp\left(1 - \frac{\bar{x}}{\lambda}\right) \geq k^*\right\} \quad (\text{generalize } \lambda_0 \text{ to } \lambda)$$

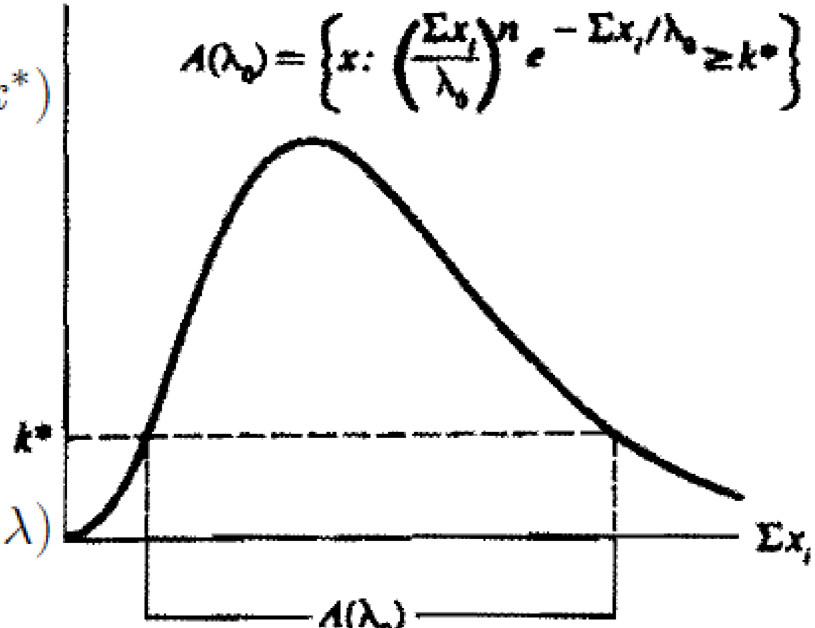
– The expression defining C(**x**) depends on **x** only through the sample mean, so the CI can be expressed in the form

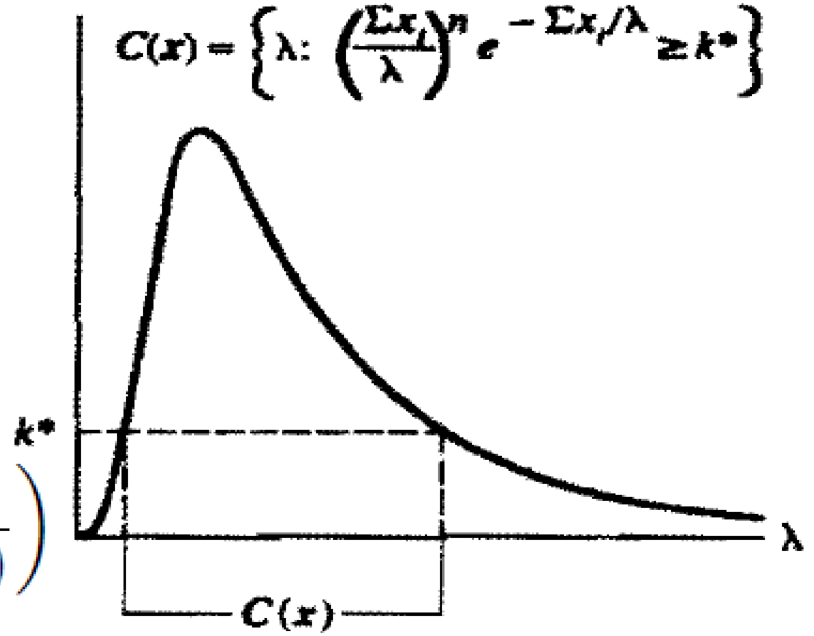
$$C(\bar{x}) = \{\lambda : L(\bar{x}) \leq \lambda \leq U(\bar{x})\}$$

– At this point L and U can be determined by imposing that Pr{x ∈ not reject} = 1-α and

$$\left(\frac{\bar{x}}{L(\bar{x})}\right)^n \exp\left(1 - \frac{\bar{x}}{L(\bar{x})}\right) = \left(\frac{\bar{x}}{U(\bar{x})}\right)^n \exp\left(1 - \frac{\bar{x}}{U(\bar{x})}\right)$$

for which the solution is numerical





TERMINOLOGY ISSUES AND BAYESIAN CIs

  - Notice that it is the CI that covers the parameter, not the opposite; in particular, our claim is not really about the probability of the parameter falling inside a CI
- This is a key remark because the random quantity, the sample statistic, is the interval, not the parameter
  - For instance, suppose a 90% CI for some parameter θ is [0.24, 0.45]
  - It is tempting to say (and many experimenters do) that "the probability is 90% that θ is in the interval [0.24, 0.45]"
  - However, such a statement is invalid since the parameter is assumed fixed and such it cannot go anywhere, it cannot fall or rise...
  - Formally, the interval [0.24, 0.45] is one of the possible realized values of the random CI for θ and, since the parameter does not move, θ is in the realized interval [0.24, 0.45] with probability 0 or 1
  - That the realized interval [0.24, 0.45] has a 90% chance of coverage, we only mean that we know that 90% of the sample points of the

Useful Notions Reviewed in This Lecture

- Let me give you a list to follow up to:
- Definition of random confidence interval/set
- Coverage and confidence of a confidence interval
- Terminology issue – what a CI really means